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Return to Equilibrium in the XY Model

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We prove that the locally perturbed XY model returns to equilibrium under the unperturbed evolution but the unperturbed model does not necessarily approach equilibrium under the perturbed evolution. In fact this latter property is false for perturbation by a local magnetization. The failure is directly attributable to the formation of bound states. If the perturbation is quadratic these problems are reduced to spectral analysis of the one-particle Hamiltonian. We demonstrate that the perturbed Hamiltonian has a finite set of eigenvalues of finite multiplicity together with some absolutely continuous spectrum. Eigenvalues can occur in the continuum if, and only if, the perturbation dislocates the system. Singular continuous spectrum cannot occur.

KEY WORDS: Return to equilibrium; XY model; quadratic perturbation.

1. INTRODUCTION

Our purpose is to investigate the return to equilibrium of the linear XY model and perturbations of this model. Various aspects of this problem have already been considered by many authors,[1-8] either by explicit calculations[1-3] or by general theory.[4-8] The most widely studied problem has been the evolution, under the unperturbed dynamics τ^0, of an equilibrium state for a locally perturbed dynamics τ. The general conclusion is that the state converges to the corresponding τ^0-equilibrium state. In Ref. 3 the reverse problem was examined, and by specific calculation it was shown that a τ^0-equilibrium state does not necessarily evolve under the perturbed evolution τ to a τ-equilibrium state. In this paper we give a general explanation of this irreversibility for a large class of quadratic perturbations. We show that it occurs if, and only if, the pertur-

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bation causes the formation of bound states. Our method is based on the scattering formalism developed in Ref. 5. The problem then reduces to the spectral analysis of the perturbed evolution on the one-particle space. We show that for local perturbations this spectrum consists of an absolutely continuous part together with a finite number of eigenvalues with finite multiplicity. Eigenvalues in the continuum can occur but then the corresponding eigenfunctions are strictly localized.

Although we mostly consider the two-sided XY model, our results are also valid for the one-sided model because it can be obtained from the two-sided model by a local perturbation.

2. PRELIMINARIES

Let $\mathscr{A}^s$ denote the C^* algebra of Pauli spin operators $\{\sigma_1^x, \sigma_2^x, \sigma_3^x; x \in \mathbb{Z}\}$ (see, for example, Ref. 6, Sec. 6.2.1) and for each finite interval $I \subset \mathbb{Z}$ and $\gamma \in \mathbb{R}$ define the XY Hamiltonian

$$H_{0,I} = -\frac{1}{4} \sum_{x \in I} \{(1+\gamma)\, \sigma_1^x \sigma_1^{x+1} + (1-\gamma)\, \sigma_2^x \sigma_2^{x+1}\}$$

It follows from general theory (e.g., Ref. 6, Sec. 6.2.1) that the norm limits

$$\tau_t^0(A) = \lim_{I \to \mathbb{Z}} e^{itH_{0,I}} A e^{-itH_{0,I}}$$

exist for all $A \in \mathscr{A}^s$ and $t \in \mathbb{R}$. The notation $I \to \mathbb{Z}$ indicates that the intervals increase and eventually contain any subinterval of $\mathbb{Z}$. The limits $\tau_t^0(A)$ define a strongly continuous one-parameter group τ^0 of * automorphisms of $\mathscr{A}$, the XY evolution.

Next, if $V = V^* \in \mathscr{A}$ and

$$H_I = H_{0,I} + V$$

then the norm limits

$$\tau_t(A) = \lim_{I \to \mathbb{Z}} e^{itH_I} A e^{-itH_I}, \qquad A \in \mathscr{A}^s,\ t \in \mathbb{R}$$

also exist. The corresponding automorphism group τ will be referred to as the perturbed evolution.

Now if $\theta: \mathscr{A}^s \to \mathscr{A}^s$ denotes the * automorphism of $\mathscr{A}^s$ such that

$$\theta(\sigma_1^x) = -\sigma_1^x, \qquad \theta(\sigma_2^x) = -\sigma_2^x, \qquad \theta(\sigma_3^x) = \sigma_3^x$$

for all $x \in \mathbb{Z}$, then $\theta(H_{0,I}) = H_{0,I}$ and consequently

$$\theta\tau^0 = \tau^0\theta$$

If $\theta(V) = V$, then it also follows that

$$\theta\tau = \tau\theta$$

Elements $A \in \mathscr{A}^s$ with the property $\theta(A) = A$ are called even and elements such that $\theta(A) = -A$ are called odd. Each $A \in \mathscr{A}$ has a unique decomposition as a sum of odd and even elements A_- and A_+ where $A_\pm = [A \pm \theta(A)]/2$. The even elements of $\mathscr{A}^s$ generate a C^* subalgebra $\mathscr{A}^s_+$ of $\mathscr{A}^s$, and the odd elements generate a Banach subspace $\mathscr{A}^s_-$ of $\mathscr{A}^s$. We only consider even perturbations V and hence both τ^0 and τ restrict to * automorphism groups of $\mathscr{A}^s_+$.

Practically all calculations on the XY model are based upon a replacement of the Pauli spin operators $\{\sigma^x_i\}$ by Fermi annihilation and creation operators $\{a_x, a^*_x; x \in \mathbb{Z}\}$. This is a standard procedure usually referred to as a Jordan–Wigner transformation. We use the general formulation given in Ref. 8, Section 2.

Let $\hat{\mathscr{A}}$ be the C^* algebra generated by $\mathscr{A}^s$ and an element T satisfying

$$T = T^*, \qquad T^2 = 1$$

and

$$TAT = \theta_-(A), \qquad A \in \mathscr{A}^s$$

where θ_- is the * automorphism of $\mathscr{A}^s$ such that $\theta_-(\sigma^x_i) = -\sigma^x_i$ if $x \leqslant 0$ and $\theta_-(\sigma^x_i) = \sigma^x_i$ if $x \geqslant 1$, for $i = 1, 2$. Then define annihilation and creation operators a_x and a^*_x by

$$a_x = T\pi_x(\sigma^x_1 + i\sigma^x_2)/2, \qquad a^*_x = T\pi_x(\sigma^x_1 - \sigma^x_2)/2 \tag{2.1}$$

where

$$\begin{aligned} \pi_x &= \sigma^1_3\sigma^2_3 \cdots \sigma^{x-1}_3 && \text{if} \quad x > 1 \\ &= 1 && \text{if} \quad x = 1 \\ &= \sigma^x_3\sigma^{x+1}_3 \cdots \sigma^0_3 && \text{if} \quad x < 1 \end{aligned}$$

One verifies that the a and a^* satisfy the anticommutation relations

$$\{a_x, a_y\} = 0 = \{a^*_x, a^*_y\}$$
$$\{a_x, a^*_y\} = \delta_{x,y}$$

and we denote by $\mathscr{A}^{CAR}$ the C^* subalgebra of $\hat{\mathscr{A}}$ generated by the a_x and a^*_y. Now θ, and also θ_-, can be extended to $\hat{\mathscr{A}}$ by setting $\theta(T) = T = \theta_-(T)$. Then defining odd and even parts of $\hat{\mathscr{A}}$ and $\mathscr{A}^{CAR}$ by

$$\hat{\mathscr{A}}_\pm = \{A \in \hat{\mathscr{A}}; \theta(A) = \pm A\}$$
$$\mathscr{A}^{CAR}_\pm = \{A \in \mathscr{A}^{CAR}; \theta(A) = \pm A\}$$

one has $\theta(a_x) = -a_x$, $\theta(a_x^*) = -a_x^*$ and furthermore

$$\mathscr{A}^s_+ = \mathscr{A}^{CAR}_+ = \mathscr{A}^{CAR} \cap \hat{\mathscr{A}}_+ = \mathscr{A}^s \cap \hat{\mathscr{A}}_+$$
$$\mathscr{A}^s_- = T\mathscr{A}^{CAR}_- = T(\mathscr{A}^{CAR} \cap \hat{\mathscr{A}}_-) = \mathscr{A}^s \cap \hat{\mathscr{A}}_-$$

Thus, the even parts of $\mathscr{A}^s$ and $\mathscr{A}^{CAR}$ coincide but the odd parts are distinct.

Since $H_{0,I} \in \mathscr{A}^s_+ = \mathscr{A}^{CAR}_+$ one can also define τ^0 on $\mathscr{A}^{CAR}$. In fact, explicit calculation with the Jordan–Wigner transformation (2.1) gives

$$H_{0,I} = \frac{1}{2} \sum_{x \in I} \{(a_x^* a_{x+1} + a_{x+1}^* a_x) + \gamma(a_x^* a_{x+1}^* + a_{x+1} a_x)\} \qquad (2.2)$$

Moreover, if $V = V^* \in \mathscr{A}^s_+$, one can also define τ on $\mathscr{A}^{CAR}$. Both τ^0 and τ then leave $\mathscr{A}^{CAR}_+$ invariant. Finally, one can extend τ^0 to $\hat{\mathscr{A}}$ by setting

$$\tau_t^0(T) = TV_{0,t}$$

where $V_{0,t}$ is the unitary operator

$$V_{0,t} = \sum_{m \geq 0} i^m \int_0^t dt_1 \cdots \int_0^t dt_m \tau_{t_m}(P_0),..., \tau_{t_1}(P_0)$$

and

$$P_0 = \lim_{I \to \mathbb{Z}} \theta_-(H_{0,I}) - H_{0,I}$$
$$= \tfrac{1}{2}\{(1+\gamma)\, \sigma_1^0 \sigma_1^1 + (1-\gamma)\, \sigma_2^0 \sigma_2^1\}$$

The perturbed group τ extends in a similar fashion, replacing $H_{0,I}$ by H_I where appropriate. (For further details see Ref. 8, or for a general discussion of perturbed automorphism groups, cocycles, etc., see Ref. 6, Section 5.4.1.)

Next we need some properties of KMS states and rely on Ref. 6 for general background.

Let $\beta \in \mathbb{R}$. Then it follows from Ref. 9 and 10 (see also Ref. 6, Theorem 6.2.47) that there exists a unique (τ^0, β)-KMS state ω_0^β, and also a unique (τ, β)-KMS ω^β, over $\mathscr{A}^s$. Moreover, there are also unique such states ρ_0^β and ρ^β over $\mathscr{A}^{CAR}$. We need the following result of Araki.[8]

Proposition 2.1. Let τ^0 denote the XY evolution and τ the perturbed evolution obtained with an even perturbation. For each $\beta \in \mathbb{R}$, the KMS states $\omega_0^\beta, \omega^\beta$ over $\mathscr{A}^s$ and ρ_0^β, ρ^β over $\mathscr{A}^{CAR}$ satisfy

$$\omega_0^\beta(A) = \rho_0^\beta(A), \qquad \omega^\beta(A) = \rho^\beta(A)$$

for all $A \in \mathscr{A}^s_+ = \mathscr{A}^{CAR}_+$.

Proof. The proof is given at the beginning of Section 4 of Ref. 8. The argument can be paraphrased as follows.

There exists a (τ^0, β)-KMS state $\hat{\omega}_0^\beta$ over $\hat{\mathscr{A}}$ obtained as a weak limit of the local Gibbs states corresponding to the $H_{0,I}$. This state can be assumed to be θ invariant and also invariant under the automorphism $\hat{\theta}_-$ of $\hat{\mathscr{A}}$ satisfying $\hat{\theta}_-(A) = A$, $A \in \mathscr{A}^{CAR}$ and $\hat{\theta}_-(T) = -T$. The restrictions of $\hat{\omega}_0^\beta$ to $\mathscr{A}^s$ and $\mathscr{A}^{CAR}$ are automatically (τ^0, β)-KMS states of $\mathscr{A}^s$ and $\mathscr{A}^{CAR}$, respectively. Hence these restrictions are equal to ω_0^β and ρ_0^β by uniqueness of these states. Therefore, $\omega_0^\beta = \rho_0^\beta$ on $\mathscr{A}_+^s = \mathscr{A}_+^{CAR}$. The argument for ω^β and ρ^β is identical.

Next, it is convenient to describe $\mathscr{A}_+^s = \mathscr{A}_+^{CAR}$ as the even part of a self-dual *CAR*-algebra. For this we adopt the conventions of Refs. 7, 8, and 11.

Let l_2 denote the Hilbert space of square summable sequences $\{f(x)\}_{x \in \mathbb{Z}}$ and l_2^* its dual. We can identify l_2^* with l_2 by setting $\{f(x)\}^* = \{\bar{f}(x)\}$ and $f^*(g) = (f, g)$. Next for $f_1, f_2 \in l_2$ define $a(f_1), a^*(f_2) \in \mathscr{A}^{CAR}$ by

$$a(f_1) = \sum_x a_x f_1(x), \qquad a^*(f_2) = \sum_x a_x^* f_2(x)$$

and for $F = f_1 \oplus f_2^* \in l_2 \oplus l_2^*$ set

$$B(F) = a^*(f_1) + a(\bar{f}_2) \tag{2.3}$$

Then one has $B(F) \in \mathscr{A}^{CAR}$ and

$$B(F)^* = B(\Gamma F)$$

where $\Gamma(f_1 \oplus f_2^*) = f_2 \oplus f_1^*$, and

$$\{B(F)^*, B(G)\} = (F, G)$$

where

$$(f_1 \oplus f_2^*, g_1 \oplus g_2^*) = (f_1, g_1) + (g_2, f_2)$$

Now $\mathscr{A}^{CAR}$ is generated by polynomials in the $B(F)$, and since $\theta[B(F)] = -B(F)$ the even algebra $\mathscr{A}_+^{CAR}$ is generated by even polynomials in the $B(F)$. Next, one calculates from (2.2) and (2.3) that

$$\lim_{I \to \mathbb{Z}} [H_{0,I}, B(F)] = B(h_0 F)$$

where

$$h_0 = \begin{bmatrix} (U + U^*)/2 & \gamma(U - U^*)/2 \\ -\gamma(U - U^*)/2 & -(U + U^*)/2 \end{bmatrix}$$

and $U\{f(x)\} = \{f(x+1)\}$. It follows that $h_0 = h_0^*$ and $\Gamma h_0 = -h_0 \Gamma$. Moreover, the action of τ^0 on $\mathscr{A}^{CAR}$ is such that

$$\tau_t^0 B(F) = B(e^{ith_0} F)$$

Alternatively introducing the Fourier transform

$$\hat{f}(\varphi) = \sum_{x \in \mathbb{Z}} f(x)\, e^{ix\varphi}$$

one has

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi\, \hat{f}(\varphi)\, e^{-ix\varphi}$$

and

$$(\widehat{Uf})(\varphi) = e^{-i\varphi} \hat{f}(\varphi), \qquad (\widehat{U^* f})(\varphi) = e^{i\varphi} \hat{f}(\varphi)$$

Consequently

$$(\widehat{h_0 F})(\varphi) = \begin{bmatrix} \cos\varphi & -i\gamma \sin\varphi \\ i\gamma \sin\varphi & -\cos\varphi \end{bmatrix} \begin{bmatrix} \hat{f}_1(\varphi) \\ \hat{f}_2(\varphi) \end{bmatrix}$$

Since the matrix on the right has eigenvalues $\pm(\cos^2\varphi + \gamma^2 \sin^2\varphi)^{1/2}$, this representation establishes that the spectrum $\sigma(h_0)$ of h_0 is absolutely continuous if $|\gamma| \neq 1$, and in fact $\sigma(h_0) = \overline{-I_\gamma \cup I_\gamma}$ where $I_\gamma = \langle |\gamma| \wedge 1, 1 \vee |\gamma| \rangle$. If $|\gamma| = 1$ then $\sigma(h_0) = \{-1, 1\}$.

Next we introduce the class of perturbations that will be analyzed in more detail in Section 3.

The perturbation $V = V^* \in \mathscr{A}_+^s = \mathscr{A}_+^{CAR}$ is called *quadratic* if

$$[V, B(F)] = B(vF)$$

where v is a bounded operator on $h = l_2 \oplus l_2^*$ satisfying $v\Gamma = -\Gamma v$. Moreover, if there is a finite interval $I \subset \mathbb{Z}$ such that $(vF)(x) = 0$ for $x \notin I$ we call V a *local quadratic* perturbation and the corresponding v a local perturbation. In particular, v is finite rank. If, more generally, v is of trace class we refer to V as a *quadratic trace class* perturbation.

Typically a quadratic perturbation V is a quadratic function of $\{\sigma_1^x, \sigma_2^x; x \in \mathbb{Z}\}$ and a local quadratic perturbation is a quadratic function of $\{\sigma_1^x, \sigma_2^x; x \in I\}$. Since $\sigma_3^x = -i\sigma_1^x \sigma_2^x$, linear combinations of σ_3^x are also quadratic perturbations. The perturbation

$$V = \frac{1}{2} \sum_{x \in \mathbb{Z}} \lambda_x \sigma_3^x$$

with $\lambda_x \in \mathbb{R}$ and

$$\|V\| = \frac{1}{2} \sum_{x \in \mathbb{Z}} |\lambda_x| < +\infty$$

is an example of a quadratic trace class perturbation, which is not necessarily local. The associated operator v has the action

$$[v(f_1 \oplus f_2^*)](x) = \lambda_x(f_1 \oplus -f_2^*)(x)$$

and hence v has finite trace norm $\|v\|_{\mathrm{tr}}$ given by

$$\|v\|_{\mathrm{tr}} = 2 \sum_{x \in \mathbb{Z}} |\lambda_x| = 4 \|V\|$$

Finally, we remark that if V is a quadratic perturbation then τ acts on $\mathscr{A}^{CAR}$ in a similar manner to τ^0. One has

$$\tau_t B(F) = B(e^{ith} F)$$

where $h = h_0 + v$. Moreover the unique (τ^0, β)- and (τ, β)-KMS states, ρ_0^β and ρ^β over $\mathscr{A}^{CAR}$, can be calculated as the quasi-free states with two-point functions

$$\begin{aligned} \rho_0^\beta(B(F)^* B(G)) &= (F, (1 + e^{\beta h_0})^{-1} G) \\ \rho^\beta(B(F)^* B(G)) &= (F, (1 + e^{\beta h})^{-1} G) \end{aligned} \tag{2.4}$$

(see, for example, Ref. 6, Example 5.3.2).

Remark. Although it follows from Proposition 2.1 that $\rho_0^\beta = \omega_0^\beta$ and $\rho^\beta = \omega^\beta$ on $\mathscr{A}_+^s = \mathscr{A}_+^{CAR}$, it is not easy to verify by direct calculation that ρ_0^β and ρ^β satisfy the KMS condition on $\mathscr{A}^s$. The difficulty arises in verifying the KMS relation

$$\rho_0^\beta(A\tau_{i\beta}^0(B)) = \rho_0^\beta(BA) \tag{2.5}$$

for odd elements $A, B \in \mathscr{A}^s$. For example, it follows from the Jordan–Wigner transformations (2.1) that

$$\begin{aligned} \sigma_1^x &= T \prod_{y=1}^{x-1} (2a_y^* a_y - 1)(a_x + a_x^*) \\ \sigma_2^z &= T \prod_{y=1}^{z-1} (2a_y^* a_y - 1)(a_z - a_z^*) \end{aligned} \tag{2.6}$$

for $x, z > 1$. Then if $A = \sigma_1^x$ and $B = \sigma_2^z$, one has $A\tau_{i\beta}^0(B) \in \mathscr{A}_+^s = \mathscr{A}_+^{CAR}$ and $BA \in \mathscr{A}_+^s = \mathscr{A}_+^{CAR}$; hence, both sides of (2.5) are well-defined. But the presence of the factor T in (2.1) means that (2.5) does not follow directly from the (τ^0, β)-KMS condition on $\mathscr{A}^{CAR}$.

3. RETURN TO EQUILIBRIUM

Our discussion of the time-development of perturbed equilibrium states, i.e. (τ, β)-KMS states, is based upon the techniques of scattering theory. The first basic result we use is the following.

Proposition 3.1. Let h_0 be a self-adjoint operator with absolutely continuous spectrum on a Hilbert space $\mathscr{h}$ and v a symmetric trace-class operator. Set $h = h_0 + v$ and let $\mathscr{h}_{\rm ac}$ denote the subspace of absolute continuity of h.

1. The strong limits

$$\lim_{t \to \pm\infty} e^{-iht} e^{ih_0 t} f = \Omega_{\pm} f$$

exist for all $f \in \mathscr{h}$ and their ranges $R(\Omega_{\pm})$ satisfy

$$R(\Omega_+) = \mathscr{h}_{\rm ac} = R(\Omega_-)$$

2. The inverse operators $\Omega_{\pm}^{-1} : \mathscr{h}_{\rm ac} \mapsto \mathscr{h}$ are given by

$$\lim_{t \to \pm\infty} e^{-ih_0 t} e^{iht} f = \Omega_{\pm}^{-1} f$$

for all $f \in \mathscr{h}_{\rm ac}$.

3. The operators $\Omega_{\pm}$ satisfy the intertwining property

$$\Omega_{\pm} e^{ih_0 t} = e^{iht} \Omega_{\pm}, \qquad t \in \mathbb{R}$$

Proofs of this result can be extracted from any of the standard references on scattering theorem, e.g., Ref. 12, Chapter X or Ref. 13, Chapter XI. The subtle part is the statement $R(\Omega_{\pm}) = h_{ac}$, which is referred to as completeness. This relies heavily on the assumption that v is trace-class.

This result has immediate implications for perturbations of the XY evolution.

Corollary 3.2. Let τ^0 and τ be strongly continuous one-parameter groups of * automorphisms of the self-dual C^* algebra $\mathscr{A}^{CAR}$ over $\mathscr{h} = l^2 \oplus l^{2*}$ and suppose

$$\tau_t^0(B(F)) = B(e^{ith_0} F), \qquad \tau_t(B(F)) = B(e^{it(h_0 + v)} F), \qquad F \in \mathscr{h}$$

where h_0 is self-adjoint with absolutely continuous spectrum and v is symmetric and of trace-class.

It follows that the norm limits

$$\gamma_{\pm}(A) = \lim_{t \to \pm\infty} \tau_{-t} \tau_t^0(A)$$

exist for all $A \in \mathscr{A}^{CAR}$ and in particular

$$\gamma_{\pm} B(F) = B(\Omega_{\pm} F)$$

where

$$\Omega_{\pm} F = \lim_{t \to \pm\infty} e^{-it(h_0 + v)} e^{ith_0} F$$

The $\gamma_{\pm}$ are norm-preserving * morphisms of $\mathscr{A}_{CAR}$ with ranges $R(\gamma_{\pm})$ equal to the C^* subalgebra of $\mathscr{A}^{CAR}$ generated by $\{B(F); F \in R(\Omega_{\pm})\}$. Moreover

$$\gamma_{\pm} \tau_t^0 = \tau_t \gamma_{\pm}, \qquad t \in \mathbb{R}$$

This follows directly from Proposition 2.1 because $\mathscr{A}^{CAR}$ is generated by $\{B(F); F \in \mathscr{k}\}$ and the mapping $F \to B(F)$ is continuous. In particular $\|B(F)\| = \|F\|$.

The existence of the maps $\gamma_{\pm}$ on $\mathscr{A}^{CAR}$ allows one to invoke many of the general results of Chapter 5, Section 5.4.1 of Ref. 6. But more can be deduced from the *XY* model by using uniqueness arguments via Proposition 2.1 as Araki[8] has already remarked.

Theorem 3.3. Let τ^0 denote the *XY* evolution on the spin algebra $\mathscr{A}^s$ and τ the evolution corresponding to a quadratic trace-class perturbation of τ^0. If $\beta \in \mathbb{R}$ and ω^β, ω_0^β denote the unique (τ^0, β)-, (τ, β)-KMS states over $\mathscr{A}^s$, then

$$\lim_{t \to \pm\infty} \omega^\beta(\tau_t^0(A)) = \omega_0^\beta(A) = \omega^\beta(\gamma_{\pm}(A))$$

for all $A \in \mathscr{A}^s$.

Proof. Let ρ_0^β and ρ^β denote the unique (τ^0, β)- and (τ, β)-KMS states over $\mathscr{A}^{CAR}$. Then ρ^β is automatically τ-invariant and hence

$$\lim_{t \to \pm\infty} \rho^\beta(\tau_t^0(A)) = \lim_{t \to \pm\infty} \rho^\beta(\tau_{-t} \tau_t^0(A))$$
$$= \rho^\beta(\gamma_{\pm}(A))$$

for all $A \in \mathscr{A}^{CAR}$ by Corollary 2.2. But now we argue that $\rho^\beta \circ \gamma_{\pm}$ satisfy the (τ^0, β)-KMS condition on $\mathscr{A}^{CAR}$ and hence $\rho^\beta \circ \gamma_{\pm} = \rho_0^\beta$ by uniqueness. The proof of the KMS property follows from the intertwining property

$$\tau_t \gamma_{\pm} = \gamma_{\pm} \tau_t^0$$

which implies that

$$\rho^\beta \circ \gamma_{\pm}(A \tau_t^0(B)) = \rho^\beta(\gamma_{\pm}(A)\, \tau_t(\gamma_{\pm}(B)))$$
$$\rho^\beta \circ \gamma_{\pm}(\tau_t^0(B)\, A) = \rho^\beta(\tau_t(\gamma_{\pm}(B))\, \gamma_{\pm}(A))$$

for each pair $A, B \in \mathscr{A}^{CAR}$. Then it follows fro the (τ, β)-KMS condition for ρ^{β} and Proposition 5.3.7 of Ref. 6 that $\rho^{\beta} \circ \gamma_{\pm}$ satisfy the (τ^0, β)-KMS condition. Thus $\rho^{\beta} \circ \gamma_{\pm} = \rho_0^{\beta}$.

Finally it follows from two applications of Proposition 2.1 that

$$\begin{aligned}\lim_{t \to \pm\infty} \omega^{\beta}(\tau_t^0(A)) &= \lim_{t \to \pm\infty} \rho^{\beta}(\tau_t^0(A)) \\ &= \rho_0^{\beta}(A) \\ &= \omega_0^{\beta}(A)\end{aligned}$$

for all $A \in \mathscr{A}_+^s = \mathscr{A}_+^{CAR}$. Since, however, ω^{β} and ω_0^{β} are even and $\gamma_{\pm}(\mathscr{A}_+^s) \subseteq \mathscr{A}_+^s$, one has

$$\omega_0^{\beta} = \omega^{\beta} \circ \gamma_{\pm} = \lim_{t \to \pm\infty} \omega^{\beta} \circ \tau_t^0$$

The situation can be quite different if one considers evolution of the unperturbed KMS state ω_0^{β} under the perturbed evolution τ. This is because $h = h_0 + v$ can have point spectrum. If the spectrum of h is absolutely continuous then the situation is reversible, because $h = h_0 - v$ and the foregoing arguments apply with τ^0, τ and ω_0^{β}, ω^{β} interchanged. But if h has point spectrum the situation is irreversible. Our next objective is to analyze in more detail the spectrum $\sigma(h)$ of those h that arise as local perturbations of the XY model. We begin with two general observations.

First, the spectrum $\sigma(h)$ is contained in $[-1 \vee |\gamma| + \|v\|, 1 \vee |\gamma| + \|v\|]$ and since $\Gamma H = -H\Gamma$ it is symmetric. Moreover, the point spectrum $\sigma_p(h)$ of h is symmetric and the eigenvalues $\pm E$ have the same multiplicity. This last statement follows because $(h - E)F = 0$ is equivalent to $(h + E)\Gamma F = 0$.

Second, remark that if $|\gamma| = 1$, then $\sigma(h_0) = \{1, -1\}$, and both eigenvalues have infinite multiplicity. Let $P_{\pm}$ denote the corresponding spectral projections. Now if V is a local perturbation, $A_E = (h_0 - E)^{-1} v$ is well-defined for all $E \in \mathbb{C}\backslash\{1, -1\}$ and compact, since v is finite-rank. It then follows easily from the analytic Fredholm theorem (see, for example, Ref. 13, Chap. VI) that $(1 + A_E)^{-1}$ exists for all $E \in \mathbb{C}\backslash(\mathscr{E} \cup \{1, -1\})$ where the exceptional set $\mathscr{E}$ is a discrete subset of $\mathbb{R}\backslash\{1, -1\}$. Moreover

$$(h - E)^{-1} = (1 + A_E)^{-1} (h_0 - E)^{-1} \tag{3.1}$$

and hence $\sigma(h) \subseteq \bar{\mathscr{E}}$. But it also follows from Fredholm theory that $E \in \mathscr{E}$ if, and only if, $A_E F + F = 0$ has a nonzero solution $F \in \mathscr{h}$. Now let P denote the finite-rank projection onto the range of v. Then, for $A_E F + F = 0$ to have a nonzero solution it is necessary that

$$\det(P + PA_E) = 0 \tag{3.2}$$

But

$$P + PA_E = P + PP_+ v(1-E)^{-1} + PP_- v(1+E)^{-1}$$

Therefore (3.2) reduces to a condition of the form $Q(E) = 0$ where Q is a polynomial. Consequently the exceptional set $\mathscr{E}$ is finite. Finally if $E \in \mathscr{E}$ the corresponding eigenspace has finite multiplicity, by Fredholm theory, and $G = (h_0 - E)^{-1} F$ is an eigenfunction of h with eigenvalues E. Thus, in summary the spectrum of h differs from that of h_0 only by a finite set of eigenvalues of finite multiplicity. This conclusion is also valid for $|\gamma| \neq 1$.

Theorem 3.4. Let v be a local perturbation of the XY Hamiltonian h_0 with $|\gamma| \neq 1$.

The spectrum $\sigma(h)$ of $h = h_0 + v$ consists of two parts:

1. absolutely continuous spectrum $\sigma_{ac}(h)$ on $\sigma(h_0)$
2. a finite set of eigenvalues $\sigma_p(h) \subset [-\|h\|, \|h\|]$ each of finite multiplicity.

Proof. The description of the point spectrum on $\mathbb{R} \backslash \sigma(h_0)$ is similar to the discussion for $|\gamma| = 1$. The family $E \in \mathbb{C} \backslash \sigma(h_0) \to A_E = (h_0 - E)^{-1} v$ is an analytic operator valued function, and h has eigenvalues of finite multiplicity in the descrete set of E for which (3.2) is satisfied. The proof that $\mathscr{E}$ is finite is slightly more complicated.

After Fourier transformation, $(h_0 - E)^{-1}$ corresponds to multiplication by a 2×2 matrix with diagonal entries $\pm \cos \varphi - E$ and off-diagonal entries $\pm i\gamma \sin \varphi$ multiplied by D_E^{-1} where

$$D_E(\varphi) = E^2 - \cos^2 \varphi - \gamma^2 \sin^2 \varphi$$

Thus the kernel R_E of $(h_0 - E)^{-1}$ is immediately computable in terms of the integrals

$$I_E(x) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, e^{ix\varphi} \, D_E(\varphi)^{-1} \tag{3.3}$$

One has

$$R_E(x) = \frac{1}{2} \begin{pmatrix} I_E(x+1) + I_E(x-1) - 2EI_E(x) & \gamma[I_E(x+1) - I_E(x-1)] \\ -\gamma[I_E(x+1) - I_E(x-1)] & -I_E(x+1) - I_E(x-1) - 2EI_E(x) \end{pmatrix} \tag{3.4}$$

But the integrals can be explicitly evaluated by contour integration. Consider the case $0 \leqslant \gamma < 1$; the case $\gamma > 1$ is similar.

If $|E|>1$ then

$$I_E(x)=\frac{1}{2\sqrt{(E^2-1)(E^2-\gamma^2)}}\,[1+(-1)^{|x|}][E_\gamma-\sqrt{E_\gamma^2-1}]^{|x|} \quad (3.5)$$

where $E_\gamma=\sqrt{(E^2-\gamma^2)/(1-\gamma^2)}$. Alternatively, if $|E|<\gamma$ then

$$I_E(x)=\frac{-i^{|x|}}{2\sqrt{(\gamma^2-E^2)(1-E^2)}}\,[1+(-1)^{|x|}](E_\gamma-\sqrt{E_\gamma^2+1})^{|x|} \quad (3.5)$$

where $E_\gamma=\sqrt{(\gamma^2-E^2)/(1-\gamma^2)}$.

Now the range $P\mathscr{h}$ of v is contained in $\mathscr{h}(I)=l^2(I)\oplus l^2(I)^*$ for some finite subset $I\subset\mathbb{Z}$, and the finite-rank operator $P+PA_E$ occurring in (3.2) can be expressed as a matrix with entries which are finite linear combinations of $I_E(x_i-x_j)$ and $I_E(x_i-x_j\pm 1)$ with $x_i, x_j\in I$. Therefore, using (3.4), one deduces that if $|E|>1$ then $\det(P+PA_E)=0$ is equivalent to a condition of the form

$$[P_0(E^2)+\sqrt{E^2-1}\,P_1(E^2)]+\sqrt{E^2-\gamma^2}\,[Q_0(E^2)+\sqrt{E^2-1}\,Q_1(E^2)]=0$$

where the P_i, Q_i are polynomials whose order is determined by v. Thus after rationalization one obtains a necessary condition for an eigenvalue $Q(E^2)=0$ with Q a polynomial. This establishes that there are at most a finite number of eigenvalues E with $|E|>1$. The argument for $|E|<\gamma$ is similar, as is the case $|\gamma|>1$.

To continue the proof we need to analyze the continuous spectrum of h. This analysis is again based on the identity (3.1), but requires more detailed information about A_E. Set $I_\gamma=\langle 1\wedge|\gamma|, 1\vee|\gamma|\rangle$.

Lemma 3.5. Let I be a finite subset of $\mathbb{Z}$ and P_I the orthogonal projection from $\mathscr{h}$ into $\mathscr{h}(I)=l^2(I)\oplus l^2(I)^*$. The matrix-valued function $E\in\mathbb{C}\backslash\sigma(h_0)\to P_I(h_0-E)^{-1}P_I$ has an analytic continuation from the upper (lower) half-plane across I_γ, or $-I_\gamma$, to the lower (upper) half-plane.

Proof. Let $\delta_x\in l^2(\mathbb{Z})$ denote the function such that $\delta_x(x)=1$ and $\delta_x(y)=0$ if $y\neq x$, and set $\delta_x^{(1)}=\delta_x\oplus 0$, $\delta_x^{(2)}=0\oplus\delta_x$. Then it suffices to establish that for each pair $x, y\in\mathbb{Z}$ and $i, j\in\{1, 2\}$ the function

$$E\in\mathbb{C}\backslash\sigma(h_0)\to(\delta_x^{(i)}, (h_0-E)^{-1}\,\delta_y^{(j)})$$

has an analytic extension of the appropriate kind. But it follows from (3.4) that this is equivalent to establishing that for each $x\in\mathbb{Z}$ the function

$$E\in\mathbb{C}\backslash\sigma(h_0)\to I_E(x)=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\, e^{ix\varphi}\,D_E(\varphi)^{-1}$$

has an appropriate extension. Again this follows by contour integration.

Specifically, one calculates that given $\varepsilon > 0$ there is a $\delta > 0$ such that if $|\gamma| \wedge 1 + \varepsilon < \operatorname{Re} E < |\gamma| \vee 1 - \varepsilon$ and $0 < \operatorname{Im} E < \delta$, then

$$I_E(x) = \frac{-i}{2\sqrt{(1-E^2)(E^2-\gamma^2)}}\,[1+(-1)^{|x|}](E_\gamma - i\sqrt{1-E_\gamma^2})^{|x|}$$

where $E_\gamma = \sqrt{(E^2-\gamma^2)/(1-\gamma^2)}$. It follows immediately that $E \to I_E(x)$ can be continued across I_γ to the lower half-plane; the other continuations follow from similar calculations.

Now we want to apply Lemma 3.5 to the discussion of A_E for $E \in -I_\gamma \cup I_\gamma$.

Let $I = \{x_1, x_2, ..., x_n\} \subset \mathbb{Z}$ be such that the range of v is contained in $\mathscr{h}(I)$ and let P_I be the orthogonal projection from $\mathscr{h}$ into $\mathscr{h}(I)$. Then it follows from Lemma 3.5 that one can continue the analytic matrix-valued function $E \in \mathbb{C}\backslash\sigma(h_0) \to P_I(H_0 - E)^{-1} v$ across the cuts I_γ and $-I_\gamma$. Hence by combination of the analytic Fredholm theorem and the above reasoning one deduces the following.

Lemma 3.6. There is a finite subset $\xi \subset -I_\gamma \cup I_\gamma$ such that $[1 + P_I(h_0 - E)^{-1} v]^{-1}$ exists as a bounded operator for all $E \in \mathbb{C}\backslash(\mathscr{E} \cup \{\pm 1, \pm\gamma\})$.

In fact, $E \in \mathbb{C}\backslash\sigma(h_0) \to [1 + P_I(h_0 - E)^{-1} v]^{-1}$ is a meromorphic function which has meromorphic continuation across I_γ and $-I_\gamma$. The residues at the poles in $\mathscr{E}$ are finite-rank operators, and for each $E \in \mathscr{E}$ there is a nonzero solution of

$$[1 + P_I(h_0 - E)^{-1} v]\, F = 0, \qquad F \in \mathscr{h}$$

If $E \in \mathscr{E}$ but $E \notin \sigma(h_0)$, then it is an eigenvalue of h with eigenfunction $G = (h_0 - E)^{-1} vF$. If, however, $E \in \mathscr{E} \cap \sigma(h_0)$, then $G \in C(\mathbb{Z}) \oplus C(\mathbb{Z})^*$ and $(h - E)\, G = 0$, but it is not clear whether $G \in \mathscr{h}$.

Now consider the inverse operators $(1 + B_E)^{-1}$ with $B_E = v(h_0 - E)^{-1}$. If $|E|$ is sufficiently large one has

$$\begin{aligned}(1+B_E)^{-1} F &= F - v\left\{\sum_{n \geqslant 0} [-P_I(h_0-E)^{-1} v]^n\right\} P_I(h_0 - E)^{-1} F \\ &= F - v[1 + P_I(h_0 - E)^{-1}]\, P_I(h_0 - E)^{-1} F\end{aligned}$$

for all $F \in \mathscr{h}$. Thus if $F \in \mathscr{h}(J)$ for some J, then $E \in \mathbb{C}\backslash\sigma(h_0) \to (1 + B_E)^{-1} F$ is a meromorphic function, with values in $\mathscr{h}$, that has a meromorphic continuation through I_γ and $-I_\gamma$ with values in $\mathscr{h}$. In fact, $(1+B_E)^{-1} F \in \mathscr{h}(I \cup J)$. This follows because if $G \notin \mathscr{h}(I \cup J)$ then

$$(G, (1+B_E)^{-1} F) = (G, F) - (G, v[1+P_I(h_0-E)^{-1} v]^{-1} P_I(h_0-E)^{-1} F)$$
$$= 0$$

because $(G, F)=0$ and $vG=0$.

Next remark that if $|E|$ is sufficiently large one has

$$(h-E)^{-1} = (h_0-E)^{-1} (1+B_E)^{-1}$$

Hence if $F \in \mathscr{h}(I)$ one obtains the identity

$$(F, (h-E)^{-1} F) = (F, (h_0-E)^{-1} (1+B_E)^{-1} F)$$

for all $E \in \mathbb{R}$, with the exception of the set $\mathscr{E} \cup \{\pm 1, \pm\gamma\}$, by analytic continuation. Moreover, the function is bounded and continuous on each subinterval. This follows from the above discussion of $(1+B_E)^{-1} F$ and Lemma 3.5. Therefore one concludes the following.

Lemma 3.7. The singular continuous spectrum $\sigma_{\text{sing}}(h)$ of h is empty.

Proof. If $\langle a, b\rangle$ is an open subinterval of $\{-I_\gamma \cup I_\gamma\}\backslash\mathscr{E}$ then h has absolutely continuous spectrum on $\langle a, b\rangle$ by Theorem XIII.20 of Ref. 13 and the foregoing discussion. But this implies $\sigma_{\text{sing}}(h) \subset \mathscr{E}$, a finite set, and hence $\sigma_{\text{sing}}(h) = \varnothing$.

Thus we have proved that the continuous spectrum of h is absolutely continuous and is contained in $\sigma(h_0) = \overline{-I_\gamma \cup I_\gamma}$, but by Proposition 3.1 the operators h_0 and h restricted to the subspace $\mathscr{h}_{ac}$ of absolute continuity are unitarily equivalent, and hence $\sigma_{ac}(h) = \sigma(h_0)$. The points of $\mathscr{E}$ contained in $\sigma(h_0)$ are possibly eigenvalues of h, and if so they must have finite multiplicity. Nevertheless this argument leaves open the question of whether the thresholds ± 1, $\pm\gamma$ are eigenvalues. To handle these points, and also to obtain additional information about the eigenvalues in $\sigma(h_0)$, we adopt a different method.

Assume the range of v is contained in $\mathscr{h}(I)$ where $I = \{x_1,..., x_n\}$ and consider the eigenvalue equation $(h-E)F=0$ on the complement of I. One has

$$[(h-E)F](x) = [(h_0-E)F](x) = 0$$

for $x \leqslant x_1 - 1$ or $x \geqslant x_n + 1$. Now, making the transformation

$g_1(x) = [f_1(x) + \bar{f}_2(x)]/\sqrt{2}$, $\bar{g}_2(x) = [f_1(x) - \bar{f}_2(x)]/\sqrt{2}$, the equations $[(h_0 - E)\,F](x) = 0$ give the two equations

$$\begin{aligned} (1+\gamma)\, g_1(x+1) + (1-\gamma)\, g_1(x-1) &= 2E\bar{g}_2(x) \\ (1-\gamma)\, \bar{g}_2(x+1) + (1+\gamma)\, \bar{g}_2(x-1) &= 2Eg_1(x) \end{aligned} \tag{3.6}$$

By substitution of one into the other, one then finds that g_1 and g_2 both satisfy

$$(1-\gamma^2)\, g_i(x+2) - 2(2E^2 - 1 - \gamma^2)\, g_i(x) + (1-\gamma^2)\, g_i(x-2) = 0 \tag{3.7}$$

These recurrence relations are readily solved. The solutions to the right of x_n are

$$g_i(x_n + 2x) = a_i \lambda^{2x} + b_i \lambda^{-2x}$$
$$g_i(x_n + 2x + 1) = c_i \lambda^{2x} + d_i \lambda^{-2x}$$

where $\lambda \in \mathbb{C}$ satisfies $|\lambda| \leqslant 1$ and a_i, b_i, c_i, d_i satisfy the boundary conditions

$$g_i(x_n) = a_i + b_i,\, g_i(x_n + 2) = a_i \lambda^2 + b_i \lambda^{-2}$$
$$g_i(x_n + 1) = c_i + d_i,\ g_i(x_n + 3) = c_i \lambda^2 + d_i \lambda^{-2}$$

The value of λ depends upon E and γ. There are various possibilities. If $0 < \gamma < 1$ one has the following

$E > 1$	then $\lambda = E_\gamma - \sqrt{E_\gamma^2 - 1}$	with $E_\gamma = \sqrt{(E^2 - \gamma^2)/1 - \gamma^2}$
$\gamma \leqslant E \leqslant 1$	then $\lambda = E_\gamma + i\sqrt{1 - E_\gamma^2}$	with $E_\gamma = \sqrt{(E^2 - \gamma^2)/1 - \gamma^2}$
$0 \leqslant E < \gamma$	then $\lambda = -iE_\gamma + i\sqrt{1 + E_\gamma^2}$	with $E_\gamma = \sqrt{(\gamma^2 - E^2)/1 - \gamma^2}$

and if E is replaced by $-E$, then λ is replaced by $-\lambda$. Similar results are true if $|\gamma| > 1$.

It follows from these calculations that if $E \in \sigma(h_0)$ then $|\lambda| = 1 = |1/\lambda|$ and g_i is square-summable on $[x_n, \infty\rangle$ if, and only if, it is identically zero. Similarly g_i is square-summable on $\langle -\infty, x_i]$ if, and only if, it is zero on this interval. Thus the eigenfunctions corresponding to eigenvalues $E \in \sigma(h_0)$ are only nonzero on $\{x_1 + 1, x_1 + 2, \ldots, x_n - 1\}$. Therefore, there are at most $x_n - x_1 - 1$ eigenvalues E, repeated according to multiplicity, in the interval $[1 \wedge |\gamma|, 1 \vee |\gamma|]$, and the same number in $[-1 \wedge |\gamma|, -1 \vee |\gamma|]$ because of symmetry. This completes the proof of Theorem 3.4.

These calculations and their analogues for $|\gamma| > 1$ also establish the following:

Corollary 3.8. Assume the range of v is contained in $\mathscr{h}(I)$ where $I = \{x_1, x_2, ..., x_n\}$ and let F be a normalized eigenfunction of h corresponding to the eigenvalue E.

If $E \in \sigma(h_0)$ then $F(x) = 0$ for $x \leqslant x_1$ and $x \geqslant x_n$.

If $E \notin \sigma(h_0)$ there exists a $c_\gamma > 0$ such that

$$|F(x)| \leqslant c_\gamma \exp\{-d_\gamma(E)|x|\}$$

for all $x \in \mathbb{Z}$ where

$$d_\gamma(E) = -\log\left|\sqrt{\left|\frac{E^2-\gamma^2}{1-\gamma^2}\right|} - \sqrt{\left|\frac{E^2-1}{1-\gamma^2}\right|}\right|$$

Now we are prepared to discuss the return to equilibrium of an unperturbed equilibrium state under the perturbed evolution. The following result describes the general situation for the simplest $A \in \mathscr{A}^s_+ = \mathscr{A}^{CAR}_+$.

Theorem 3.9. Adopt the assumptions of Theorem 3.3 but further assume that v is a local perturbation. Let $\mathscr{h} = \mathscr{h}_p \oplus \mathscr{h}_{ac}$ where $\mathscr{h}_{ac}$ denotes the subspace of absolute continuity of h and $\mathscr{h}_p = \mathscr{h}_{ac}^\perp$ is the subspace of pure point spectrum of $\mathscr{h}$. Then

$$\lim_{T \to \pm\infty} \frac{1}{T} \int_0^T dt\, \omega_0^\beta(\tau_t(B(F)^*\, B(G)))$$

$$= \omega^\beta(B(F)^*\, B(G)) + \sum_j \omega_0^\beta(B(P_j F)^*\, B(P_j G)) - \omega^\beta(B(P_j F)^*\, B(P_j G))$$

where P_j are the finite-rank projections onto the eigenspace $\mathscr{h}_j \subset \mathscr{h}_p$ of h.

Proof. Let $F = F_p + F_c$ and $G = G_p + G_c$, where F_p, G_p are the components of F, G, in $\mathscr{h}_p$ and F_c, G_c the components in $\mathscr{h}_{ac}$. Then

$$\begin{aligned}\lim_{t \to \pm\infty} \omega_0^\beta(\tau_t(B(F_c)^*\, B(G_c))) &= \omega_0^\beta(B(\Omega_\pm^{-1} F_c)^*\, B(\Omega_\pm^{-1} G_c)) \\ &= \omega^\beta(B(F_c)^*\, B(G_c))\end{aligned}$$

by Statement 2 of Proposition 3.1 and a reversal of the argument used to prove Theorem 3.3. Next

$$\omega_0^\beta(B(F)^*\, B(G)) = (F, SG)$$

with $S = (1 + \exp\{\beta h_0\})^{-1}$ and hence

$$\omega_0^\beta(\tau_t(B(F_p)^*\, B(G_c))) = \sum_j e^{-itE_j}(SP_j F, e^{ith} G_c)$$

where E_j is the eigenvalue of h on $\mathscr{h}_j = P_j \mathscr{h}$. But h has absolutely continuous spectrum on $\mathscr{h}_{ac}$ and $G_c \in \mathscr{h}_{ac}$, so each term in the sum on the right converges pointwise to zero as $t \to \pm\infty$ by the Riemann–Lebesgue Lemma. Therefore since the sum is finite, by Theorem 3.4

$$\lim_{t \to \pm\infty} \omega_0^\beta(\tau_t(B(F_p)^* B(G_c))) = 0$$

Similarly

$$\lim_{t \to \pm\infty} \omega_0^\beta(\tau_t(B(F_c)^* B(G_p))) = 0$$

Finally one has

$$\omega_0^\beta(\tau_t(B(F_p)^* B(G_p))) = \sum_{j,k} e^{-i(E_j + E_k)t} \omega_0^\beta(B(P_j) F^* B(P_k G))$$

and it follows readily that

$$\lim_{T \to \pm\infty} \frac{1}{T} \int_0^T dt\, \omega_0^\beta(\tau_t(B(F_p)^* B(G_p))) = \sum_j \omega_0^\beta(B(P_j F)^* B(P_j G))$$

Combining these results and making a minor rearrangement, one obtains the desired statement.

Note that the existence of point spectrum means that the function $t \to \omega_0^\beta(\tau_t(B(F)^* B(G)))$ does not generally have a pointwise limit as $t \to \pm\infty$. The point spectrum contributes a periodic, or almost periodic, behavior. Furthermore, the ergodic average of the function is not equal to $\omega^\beta(B(F)^* B(G))$. If P denotes the orthogonal projection on $\mathscr{h}$ with range $\mathscr{h}_p$, then the difference between these quantities can be estimated from Proposition 3.4. One obtains, for example

$$\lim_{T \to \pm\infty} \left| \frac{1}{T} \int_0^T dt\, \omega_0^\beta(\tau_t(B(F)^* B(F))) - \omega^\beta(B(F)^* B(F)) \right| \leqslant 2(F, PF)$$

This shows that the difference is small if F is localized far away. For example, it follows from Theorem 3.4 that P has finite rank and it also follows fro Corollary 3.8 that the eigenfunctions of h are either strictly localized or exponentially decreasing. Thus, if $F(y) = 0$ for $|y| < R$, there exist $c, d > 0$ such that

$$(F, PF) \leqslant c e^{-dR} \|F\|^2$$

Therefore, the unperturbed equilibrium state returns to equilibrium under the perturbed evolution in a very good approximation on observables far from the center of perturbation.

4. LOCAL MAGNETIZATION

In this section we consider perturbations

$$V=\frac{1}{2}\sum_{i=1}^{n}\lambda_{x_i}\sigma_3^{x_i}, \qquad \lambda_{x_i}\neq 0$$

corresponding to the addition of a local magnetic field of strength $\lambda_{x_i}/2$ at point x_i. One readily calculates that the corresponding operator v on $\hbar$ has the action

$$[v(f_1\oplus\bar{f}_2)](x)=\sum_{i=1}^{n}\lambda_{x_i}\delta_{x_i,x}[f_1(x_i)\oplus -\bar{f}_2(x_i)]$$

Theorem 3.4 can in this case be made more precise.

Proposition 4.1. If $h=h_0+v$ where h_0 is the XY Hamiltonian and v the above local magnetization, then $\sigma(h)$ consists of two parts

1. absolutely continuous spectrum $\sigma_{ac}(h)=\sigma(h_0)$
2. a finite set of eigenvalues $\sigma_p(h)$ of finite multiplicity with $\sigma_p(h)\subset[-\|h\|, \|h\|]\backslash\sigma(h_0)$.

Proof. In light of Theorem 3.4, it suffices to prove there are no eigenvalues $E\in\sigma(h_0)$. But it follows from Corollary 3.8 that such eigenvalues exist if, and only if, the corresponding eigenfunctions F vanish for $x\leqslant x_1$ and $x\geqslant x_n$. But then the eigenvalue equation gives

$$\begin{aligned}0=[(h-E)\,F](x_1)&=[f_1(x_1+1)+\gamma\bar{f}_2(x_1+1)]/2\\&=[-\gamma f_1(x_1+1)-\bar{f}_2(x_1+1)]/2\end{aligned}$$

Therefore $f_1(x_1+1)=0=f_2(x_1+1)$ and by iteration $F=0$. Thus there are no nonzero eigenfunctions corresponding to $E\in\sigma(h_0)$.

The eigenvalues and eigenfunctions of h can be computed explicitly in simple cases. For example, if $V=\lambda\sigma_3^0/2$ and $0\leqslant\gamma<1$, then the eigenvalue equation $(h-E)\,F=0$, written in the form

$$F+(h_0-E)^{-1}\,vF=0$$

and evaluated at $x=0$, gives the eigenvalue conditions

$$\left(1-\frac{\lambda E}{\sqrt{(E^2-1)(E^2-\gamma^2)}}\right)f_1(0)=0=\left(1+\frac{\lambda E}{\sqrt{(E^2-1)(E^2-\gamma^2)}}\right)\bar{f}_2(0)$$

for $|E|>1$ and

$$\left(1+\frac{\lambda E}{\sqrt{(1-E^2)(\gamma^2-E^2)}}\right)f_1(0)=0=\left(1-\frac{\lambda E}{\sqrt{(1-E^2)(\gamma^2-E^2)}}\right)\bar{f}_2(0)$$

for $|E|<\gamma$. Here we have used (3.4) and (3.5).

Hence if $\lambda>0$ there is a solution with $E>1$ and a second solution with $-\gamma<E<0$, both such that $f_2(0)=0$. There are also the symmetric solutions $E<-1$ and $0<E<\gamma$ with $f_1(0)=0$. In particular, eigenvalues can occur in the gap in $\sigma(h_0)$.

One concludes that the irreversible phenomena described by Theorem 3.9 occur for a local magnetization. The unperturbed equilibrium state behaves periodically under the perturbed evolution but its ergodic average approaches a limit. This limit differs from the perturbed equilibrium state but the difference is exponentially small as $x\to\infty$.

5. THE ONE-SIDED *XY* MODEL

Although the foregoing discussion has been for the two-sided XY model the results also extend to the one-sided model because it can be obtained from the two-sided model by a local perturbation.

The one-sided model is defined in the manner of Section 2 but one only considers the C^* algebra of Pauli spin operators σ_i^x with $x\geqslant 1$ and the XY Hamiltonians $H_{0,I}$ with $I\subset\mathbb{Z}_+$. Alternatively one can consider the algebra $\mathscr{A}^s$ and two one-sided models, the left and the right, defined through $H_{0,I}$ with $I\subset\mathbb{Z}\backslash\mathbb{Z}_+$ and $I\subset\mathbb{Z}_+$. The description of these models is very similar to that of the two-sided model. The dynamics are defined through Hamiltonians $h_{0,l}$ and $h_{0,r}$, which act on the subspaces $\mathscr{h}(\mathbb{Z}\backslash\mathbb{Z}_+)$ and $\mathscr{h}(\mathbb{Z}_+)$ of $\mathscr{h}$, respectively. Then the Hamiltonian h_0 of the two-sided model is given by

$$h_0=h_{0,l}+h_{0,r}+v_0 \tag{5.1}$$

where v_0 is the local perturbation associated with the dislocation perturbation

$$V_0=\tfrac{1}{2}\{(a_0^*a_1+a_1^*a_0)+\gamma(a_0^*a_1^*+a_1a_0)\}$$

The one-sided XY model differs from the two-sided model in one important respect. The Hamiltonians $h_{0,r}$ or $h_{0,l}$ have absolutely continuous spectrum on $-\overline{I_\gamma}\cup\overline{I_\gamma}$, but if $\gamma\neq 0$ there is also an eigenvalue $E=0$ (see, for example, Ref. 7, Sec. 6). This can also be verified by remarking that the recurrence relations (3.6) have a solution with $E=0$ given by $g_i(x)=0$ for $x \leqslant 0$, $g_i(2x)=0$ for $x \geqslant 1$, and

$$g_1(2x+1)=\left(\frac{\gamma-1}{\gamma+1}\right)^x, \qquad g_2=0, \qquad \text{if} \quad \gamma>0$$

$$g_2(2x+1)=\left(\frac{\gamma+1}{\gamma-1}\right)^x, \qquad g_1=0, \qquad \text{if} \quad \gamma<0$$

The presence of this eigenvalue means the proofs of Theorem 3.4 and Proposition 4.1 cannot be simply repeated with h_0 replaced by $h_{0,r}$ or $h_{0,l}$. Nevertheless the analogous results can be deduced from the representation (5.1). If v is a local perturbation of $h_{0,r}$, i.e., if v acts locally on $\mathscr{h}(\mathbb{Z}_+)$, then

$$h_{0,l}+h_{0,r}+v=h_0+v-v_0$$

Now $v-v_0$ is a local perturbation of h_0 and hence Theorem 3.4 holds for $h=h_{0,l}+h_{0,r}+v$. Then, by restriction to $\mathscr{h}(\mathbb{Z}_+)$, one concludes that $h_{0,r}+v$ has at most a finite number of eigenvalues with finite multiplicity and in addition absolutely continuous spectrum in $\sigma(h_0)$. It is an easy exercise, however, to prove that the continuous spectrum is in fact equal to $\sigma(h_0)$.

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